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Reverse geometric engineering of singularities

David Berenstein

School of Natural Sciences, Institue of Advanced Study
Einstein Drive, NJ 08540, USA
E-mail: dberens@ias.edu

Abstract: One can geometrically engineer supersymmetric field theories theories by placing D-branes at or near singularities. The opposite process is described, where one can reconstruct the singularities from quiver theories. The description is in terms of a non-commutative quiver algebra which is constructed from the quiver diagram and the superpotential. The center of this non-commutative algebra is a commutative algebra, which is the ring of holomorphic functions on a variety V. If certain algebraic conditions are met, then the reverse geometric engineering produces V as the geometry that D-branes probe. It is also argued that the identification of V is invariant under Seiberg dualities.

Keywords: D-branes, Brane Dynamics in Gauge Theories, Differential and Algebraic Geometry, Non-Commutative Geometry.

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Contents

1. Introduction

String theory has solutions at weak coupling that correspond to propagation on geometrically singular spaces. In particular, one can consider Calabi-Yau compactifications that correspond to an $\mathcal{N} = (2,2)$ superconformal field theory whose target space geometric interpretation is of strings propagating on a singular Calabi-Yau manifold with a constant dilaton and without RR backgrounds.

However, although the naïve geometric interpretation is of strings propagating on a singular space, the world-sheet conformal field theory is non singular: the singularity is resolved by a stringy mechanism.

General world-sheet conformal field theories are very hard to analyze, so instead one can hope that point-like D-brane probes give a good account of the geometry and give some new notion of geometry where the space is smooth. In [1] it was proposed that the natural D-brane notion of smoothness of this space is given in terms of a *regular* non-commutative geometry and in essence this non-commutative geometry gives us a resolution of the commutative singularities.

Once we have a D-brane probe near the singularity, we can take the limit of large volume Calabi-Yau and we can then take an $\alpha' \to 0$ decoupling limit, so we are left over with a super-symmetric field theory on the world-volume of the D-brane, which has $\mathcal{N} = 1$ supersymmetry. The precise form of this $\alpha' \to 0$ limit is the main subject of the AdS/CFT correspondence, and here it is interpreted as taking the low energy effective field theory of the D-brane on it's moduli space of vacua. By this token, the theory does not have to be renormalizable, but we will also ignore the Kähler potential of the theory, so we will only be interested in holomorphic information.

This idea provides a connection between (singular) complex geometry and supersymmetric field theories and it is usually called geometric engineering. This is, given a singularity, one can construct field theories associated to it. However, for most singularities it is not known how to extract the field theory that describes the singularity. A proposal has been worked out for the specific examples of orbifolds [2], orbifolds with discrete torsion [3, 4, 5], toric singularities (without discrete torsion) [6, 7, 8] and for one parameter families of resolved ALE singularities [9, 10, 11], of which a special example is the conifold [12].

A few things are known about this process, at least for the type-II string theory. If we have a Calabi-Yau threefold singularity $\mathcal{X}$, then placing a collection of N D3-branes near the singularity produces a gauge field theory with gauge group $\prod_i \mathrm{U}(NN_i)$, with matter transforming as the $(NN_i, N\bar{N}_j)$ for some gauge groups, and with a superpotential of single trace type. If we allow the NN_i to be arbitrary integer numbers then we say we have a configuration of fractional branes.

The single trace property can be understood from the string world-sheet point of view as having just one boundary on the worldsheet. Indeed, the contribution of multi-boundary worldsheets is suppressed by the string coupling which we formally take towards zero. Perturbative non-renormalization theorems will prevent us from generating a multi-trace superpotential from loop diagrams, and can only be generated nonperturbatively. The reconstruction of the geometry depends only on the classical superpotential, this is, we do not need to have D3-branes near the singularity, any lower-dimensional D-brane obtained by dimensional reduction of the theory is just as good for reconstructing the geometry.

One also expects that the classical moduli space of the D-branes is given by the symmetric product $\mathrm{Sym}^N(\mathcal{X})$ so long as there are no RR backgrounds which can produce potential terms for D-branes propagating on X. This type of potential generically localizes the branes on a submanifold and can give rise to Myers type effects [13] which makes the D-branes into extended objects (and therefore non-local probes) as opposed to point-like probes of the geometry. So, in the absence of RR backgrounds, we can recover $\mathcal{X}$ from the moduli space of D-branes. It is also in this case that world-sheet supersymmetry is unbroken, and that one can have a topologically twisted $\mathcal{N} = 2$ topological string theory. This topologically twisted theory can compute the superpotential for a collection of D-branes.

Similarly, one can ask the opposite question: given a field theory that admits a large-N limit, with matter transforming in bifundamentals, a single-trace type superpotential, can we recover such a singular space $\mathcal{X}$ from this data? This is the question that we will ask, and we will call the process of finding such an $\mathcal{X}$ reverse geometric engineering, being the opposite of the procedure described in [14]. This is in the sense that we usually engineer field theories by putting D-branes on singularities and ask what the low energy limit of the D-branes near the singularity corresponds to.

The paper is organized as follows:

In section 2 we provide an algebraic procedure for producing the space $\mathcal{X}$ from the quiver data, and the conditions under which $\mathcal{X}$ is expected to be the right answer to the reverse geometric engineering problem.

In section 3 we give a set of examples from families of deformed A_n singularities. We explicitly compute the center of the algebra in full detail and reproduce the moduli space of vacua with a very compact calculation.

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Next, in section 4 we discuss a simple non-toric singularity, which is not an orbifold either. Here we see that the techniques presented can make sense for field theories which are interesting for the AdS/CFT correspondence, by performing marginal deformations of the superpotential of known theories.

In section 5 we show in a particular example that the geometric space $\mathcal{X}$ is well defined even if one performs Seiberg dualities on the nodes. We argue that this feature is generic: the extracted geometry does not depend on which dual realization of a field theory one is using, that in some sense the variables in the center of the algebra are gauge invariant, and therefore they do not transform under field theory dualities.

2. A non-commutative algebra for each quiver diagram

Consider a quiver diagram with a finite number of vertices V_i, $i = 1, \dots, n$ and (directed) arrows ϕ^a_{ij}, where the subindex labels indicate the vertices where the arrows begin and end. This also includes the possibilities of arrows that begin and end on the same node. The data of the quiver theory includes a choice of superpotential.

Usually when we consider a field theory associated to a quiver diagram we specify a gauge group $\mathrm{U}(N_i)$ for each vertex of the diagram, and a chiral multiplet for each arrow ϕ^a_{ij} which transforms in the $(N_i, \bar{N}_j)$ of the gauge fields at both ends. If the arrow begins and ends in the same node, this is an adjoint superfield. In our case, the N_i will be considered as arbitrary numbers, which can be set to zero if we want to and we will consider the full family of field theories associated to a given quiver diagram with arbitrary N. We will assume that the quiver diagram contains as vertices all of the topologically distinct fractional branes that appear at a singularity. If this condition is not met, then we will call the quiver diagram incomplete. However, this is a test that is done a posteriori if the procedure for finding the geometry where the branes propagate fails. One should expect in general that if one is given a set of conformal field theories labeled by N, or a set of theories labeled by N that has a conformal UV fixed point, then the quiver diagram is complete.

The superpotential will be of the form $\sum_{[\alpha]} \lambda_{[\alpha]} \operatorname{tr}(\phi^{[\alpha]})$, where $[\alpha]$ is a multi-index in the variables a, such that for any two consecutive indices in the operator, the arrow ϕ^{a_i} ends at the same vertex than where the arrow $\phi^{a_{i+1}}$ begins, and we contract the gauge indices at the vertex between these two fields. This property is true also when we take into account the cyclic property of the trace as well.

This is, for each closed loop in the quiver diagram (modulo rotations of the initial vertex) there is an associated superpotential term that we can consider adding, but in the end, for most of the interesting theories there will only be a finite number of λ which are different from zero.

It is clear that the important aspect of the gauge structure is that it lets us multiply arrows that end at a vertex V_i, with arrows that begin at that vertex;and the new composite meson field is also an arrow which transforms as the $(N, \bar{N})$ under two different gauge groups. Notice that if we want to interpret the arrows as matrices operating on vectors by left multiplication, the space on which one operates is the second index of the arrow,

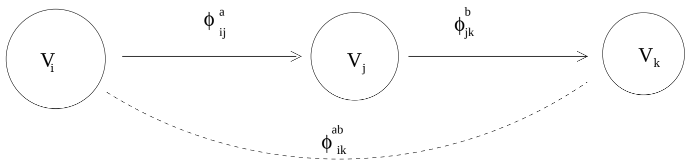

Figure 1: Composite meson field in the quiver diagram: $\phi^{ab}_{ik} = \phi^a_{ij}\phi^b_{jl}$.

and the space one lands in is associated to the first index.[1] We can take actually any composite meson field made with an arbitrary number of arrows and we see that it also has this property. We can thus consider representing each fundamental field or meson field by a square matrix whose indices are all the possible gauge indices of the different gauge groups, and where the matrix associated to the field ϕ^a_{ij} has entries only in the $[ij]$ block of this big matrix. The procedure is described in figure 1.

Any composite meson field as we described can thus be made out of multiplying and adding these types of matrices together, and will also be a matrix in these same conventions. We can call this formal concatenation of symbols a $\star$ operation, and it can be seen that it is an associative multiplication of matrices once we take into account the contraction of the gauge indices. The equations of motion derived from the superpotential are also elements of this algebra. We use these equations of motion as relations in the algebra. In this way, the algebra encodes the information of the moduli space of vacua.

As such, we have a formal algebra of matrices generated by the arrows in the quiver, and with relations given by the superpotential equations of motion. We want the gauge group to be associated to matrices in this algebra as well, but we need to project it so that it lies only along the block diagonal elements of the matrix.

To do this, it is convenient to introduce a projector for each of the vertices P_i, such that $P_i^2 = P_i$, and such that the gauge group is generated by matrices of the form

$$G = \sum_i P_i G P_i\,. \tag{2.1}$$

Similarly, the fields ϕ^a are such that

$$\phi^a_{ij} = P_i\phi^a P_j = P_i\phi^a_{ij} = \phi^a_{ij}P_j \tag{2.2}$$

so these projectors serve to keep track of the blocks in the matrices. Notice also that $P_iP_j = \delta_{ij}P_j$, so the projectors are mutually orthogonal. These projectors are not considered as truly dynamical variables, because they can only have eigenvalues $0, 1$ and not arbitrary complex numbers.

Notice that $N_i = \mathrm{tr}(P_i)$, so we can measure the rank of the gauge groups by taking traces of these projectors, so they do measure some of the discrete degrees of freedom associated to the choices we make on a quiver diagram.

[1] Fundamental arrows starting at j and ending at i represent generators of the group $ext^1([i],[j])$ for the coherent sheaves (D-branes) associated to nodes i, j. This is the opposite convention for the arrows than [1]. See also [15] for how the ext groups appear in the discussion.

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The meson algebra $\mathcal{A}$ associated to the quiver diagram will be generated by the chiral fields and the projectors. It is the algebra of meson fields, and it is an associative algebra by construction. The relations in this algebra are given by the fact that the projectors are mutually orthogonal, by the superpotential equations of motion and by eqs. (2.2) which specify where the arrows begin and end. The variables in the algebra are precursors of gauge invariant operators. To produce a gauge invariant operator we just need to take a trace of the element in the algebra.

To solve for a point in the moduli space of vacua is then to look for solutions to the superpotential equations of motion where we have specific matrices for all the arrows and projectors. This amounts to finding a representation of the non-commutative algebra as given above, with the projectors included, up to gauge equivalence. Notice that the main role of the projectors is to reduce the gauge group from the one of $n \times n$ matrices to the smaller group that commutes with all of the P_i, and to give the algebra an identity operator $1 = \sum_i P_i$.

In this way we can take the moduli space problem and phrase it in terms of representation theory of a generically non-commutative algebra. The algebra is non-commutative any time we have more than one node as it is easy to see that

$$P_1 \phi_{12} = \phi_{12} \neq \phi_{12} P_1 = 0\,. \tag{2.3}$$

The physical interpretation now is that this non-commutative algebra is associated to a non-commutative (algebraic) geometry that D-branes see, and we need to extract the closed string target space out of this non-commutative geometry. Once we have this non-commutative interpretation of the quantum field theory, the techniques developed in [1, 16, 17, 18] can be utilized to analyze the field theory moduli space.

An important subalgebra of the non-commutative algebra $\mathcal{A}$ is the center of the algebra, $\mathcal{Z}\mathcal{A}$, which corresponds to a commutative geometry. When the center of the algebra is big enough (we will make this notion precise later), this commutative geometry gives rise to some singular algebraic geometry which turns out to be of dimension 3 for most interesting superconformal models.

The key point of this commutative algebra is the idea exposed in the Seiberg-Witten [19] approach to non-commutative field theory. There are two geometries: one non-commutative for open strings (D-branes), and a commutative geometry for closed strings. In the present case, the commutative geometry for closed strings will correspond to the algebraic geometry associated to the center of the quiver algebra. We will call this variety V.

One then needs to show that the moduli space of the D-branes is essentially the symmetric space $\mathrm{Sym}^N(V)$. The fact that the moduli space is some symmetric space follows from the statement that the direct sum of two representations of an associative algebra gives a representation of the algebra [16]. Since one can choose the matrices associated to the direct sum representations to have block diagonal form, one can have an interpretation of the configuration in terms of two objects on a given space, as in matrix theory [20]. One needs to show that the space of irreducible representations (which are the non-commutative analog of points) away from the singularities is V, and that at the singularities there are additional

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fractional brane representations. The trick that lets us relate the irreducible representations to V is Schur's lemma. It tells us that on an irreducible representation of an algebra in terms of $n\times n$ matrices, any element of the center of the algebra is proportional to the identity. In this way we have an evaluation map from non-commutative points to the variety V.

If $\mathcal{A}$ is finitely generated as a module over $\mathcal{ZA}$, then the irreducible representations of $\mathcal{A}$ have bounded dimension. Consider s_α, $\alpha = 1,\dots,n$ a basis of $\mathcal{A}$ as a module over $\mathcal{ZA}$, where we can take $s_1 = 1$ if we want to. Then the relations in the algebra have to read

$$s_\alpha s_\beta = \sum_\gamma f_{\alpha\beta\gamma}(\mathcal{ZA}) s_\gamma\,. \tag{2.4}$$

Given a vector v in the representation of the algebra, the elements $s_\alpha v$ generate the representation, as the $f_{\alpha\beta\gamma}$ are given by constant numbers.

These types of algebras guarantee that a bulk D-brane is made out of finitely many fractional branes, and it is usually the case that when this condition is met there are enough representations in the algebra to cover V completely, as we will see in the examples.

If this finitely generated condition is not met, then generically one can not cover the variety V with finite-dimensional representations of the quiver, and therefore the moduli space of vacua does not behave like $\mathrm{Sym}^n(V)$ for any n. This is the statement that the center of the algebra is not large enough. Hence we will ask that the quiver algebra is indeed finitely generated over the center in order to make sense of the geometry of V from the moduli space of vacua.

In the paper [21] it was explicitly verified that these non-commutative techniques reproduced the conifold geometry exactly for various distinct constructions of the non-commutative algebra, including the fractional branes at the singularity.

To summarize the construction, here is the recipe for producing the reverse geometric engineered complex geometry.

1. Give a connected quiver diagram with some choice of superpotential, such that there is at least one set of integers $N_i > 0$ for which the gauge theory associated to the quiver with gauge group $\mathrm{U}(NN_i)$ in vertex V_i is consistent in four dimensions (the anomaly factorizes). This is necessary to have a bulk D-brane.

2. Construct the quiver algebra $\mathcal{A}$ with generators given by the chiral fields and with a projector associated to each node. The relations in the generators are given by the superpotential equations of motion and by the location of arrows in the quiver diagram.

3. Out of $\mathcal{A}$ extract the center of the algebra $\mathcal{ZA}$. Verify that $\mathcal{ZA}$ correspond to the ring of holomorphic functions on a variety of complex dimension 3, which we will call V. For this center to be interesting it is clear that we need to have enough relations in the quiver algebra to move any arrow from the left of an element of the center to the right. In particular this implies that every arrow in the quiver diagram needs to appear in at least one term in the superpotential.

4. Verify that for each point in V away from the singularities of V there is one unique irreducible representation of the algebra $\mathcal{A}$. Usually this is accomplished only if the algebra $\mathcal{A}$ is finitely generated over $\mathcal{Z}\mathcal{A}$. In brane language, this tells us that a bulk brane is made out of finitely many fractional branes, so we will require this property as a condition to check.

5. If all of the above conditions are met, we say that the field theory corresponds to the dynamics of point like (and fractional) D-branes on V.

Important Remark. The construction above provides a purely algebraic geometric background. For other models it might be the case that it is still possible to produce AdS/CFT duals of the theories (see [5] for example), but these might involve turning on RR and NS two forms. For these models the RR potentials obstruct the moduli space of the D-branes so that point like branes can not explore the commutative geometry in it's entirety. So, although these quiver theories are consistent field theories, they lead to a model which can not be interpreted in terms of a $(2,2)$ sigma model on the worldsheet. A second obstruction to the above process of defining a commutative geometry is that we might have some set of branes in a quiver diagram which are extended when the target space is compact. If these extended branes are such that they can not all be shrunk to zero geometrical size simultaneously, then the idea that we have all of the fractional branes at a singularity is not valid either. The rest of the paper will deal with examples where the conditions for finding the geometry of the variety V are satisfied.

The second point worth noticing, is that the procedure described above is not an algorithm. The reason for this is that there is no known algorithm to calculate the center of an abstract algebra. The ability to calculate the center effectively depends on how tractable the structure of the quiver algebra is.

3. Families of resolved A_{n-1} singularities

Here we will consider the affine quiver diagrams of an A_{n-1} singularity with a deformed superpotential, as has appeared in [11]. See also [22].

The quiver diagram is as shown in the figure 2. We have n nodes, V_i, $i = 1, \dots, n$. Three families of fields$x_{i,i+1}$, $y_{i,i-1}$ and $z_{i,i}$.

The superpotential is given by

$$\sum_i \mathrm{tr}(z_{ii}x_{i,i+1}y_{i+1,i} - z_{ii}y_{i,i-1}x_{i-1,i}) - \mathrm{tr}(Q_i(z_{i,i})) \quad (3.1)$$

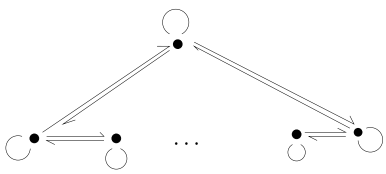

Figure 2: A_{n-1} quiver diagram.

and $i = i'$ mod (n) when the index falls outside $1, \dots, n$, and with the β_i polynomials of degree less than or equal to k for some k,and such that $\sum_i Q_i(z) = 0$, if we replace all the variables $z_{i,i}$ by a single variable z. The polynomials $Q_i(z)$ are of arbitrary degree, and there is one such polynomial for each node in the quiver diagram.

The quiver algebra has projectors P_i for $i = 1, \ldots, n$, and is generated by the $x_{i,i+1}$, $y_{i,i-1}, z_{ii}$ with additional constraints

$$\begin{aligned} x_{i,i+1}y_{i+1,i} - y_{i,i-1}x_{i-1,i} - Q_i'(z_i) &= 0\,, \\ z_{i,i}x_{i,i+1} - x_{i,i+1}z_{i+1,i+1} &= 0\,, \\ y_{i+1,i}z_{i,i} - z_{i+1,i+1}y_{i+1,i} &= 0 \end{aligned} \tag{3.2}$$

and the projection equations $P_j x_{i,i+1} = \delta_{i,j} x_{i,i+1}$ etc.

Consider the new variables

$$\begin{aligned} z &= \sum_i z_{ii}\,, \qquad x = \sum_i x_{i,i+1}\,, \\ y &= \sum_i y_{i+1,i}\,, \qquad \sigma = \sum_i P_i \eta^i \end{aligned} \tag{3.3}$$

for $\eta = \exp(2\pi i/n)$ a primitive n-th root of unity.

It is easy to see that we can recover the P_i from σ when we consider the n monomials σ^k for $k = 1, \ldots, n$. Namely

$$P_k = \frac{1}{n} \sum_j \eta^{-kj} \sigma^j\,. \tag{3.4}$$

From here we can also recuperate the individual $x_{i,i+1} = P_i x$, and similarly for $y_{i+1,i}$, $z_{i,i}$, so we have a new set of generators of the algebra over $\mathbb{C}$. These are x, y, z, σ.

In these new variables, the relations read

$$\begin{aligned} \sigma z &= z\sigma\,, \qquad \sigma x = \eta x \sigma\,, \\ \sigma y &= \eta^{-1} y \sigma\,, \qquad xz = zx\,, \\ yz &= zy\,, \qquad xy - yx = \sum_{k=1}^{n-1} \tilde{Q}_k(z)\sigma^k\,. \end{aligned} \tag{3.5}$$

The condition $\sum_k Q_k = 0$ is necessary so that on the right hand side there is no term proportional to $\sigma^n = 1$. In the expression above $\tilde{Q}$ are some new linear combinations of the derivatives of Q'. It follows that

$$Q_j' = P_j \sum_{k=1}^{n-1} \tilde{Q}_k(z)\sigma^k P_j = \sum_{k=1}^{n-1} \tilde{Q}_k(z_{jj})\eta^{kj}\,. \tag{3.6}$$

It is clear from the equations above that z is in the center of the algebra. It is also an easy matter to check that $u = x^n, v = y^n$ are elements in the center of the algebra. This is necessary in order to commute with σ, and the commutator with x or y is zero from identities obtained from sums over roots of unity.

There is one additional element of the center. Notice that xy commutes with σ, but it does not commute with x, y. Indeed

$$(x)xy - xy(x) = x(xy - yx) = x \sum_k \tilde{Q}_k(z)\sigma^k\,. \tag{3.7}$$

But we also see that $x\sigma^k = \sigma^k\eta^{-k}x$, so each of the terms in the right hand side can be written as a commutator

$$x\sigma^k = \left[x, \sigma^k/(1-\eta^k)\right] \tag{3.8}$$

and we have

$$\left[x, xy - \sum_k \tilde{Q}_k(z)\sigma^k/(1-\eta^k)\right] = 0\,. \tag{3.9}$$

We can call this variable w,

$$w = xy - \sum_k \tilde{Q}_k(z)\sigma^k/(1-\eta^k)\,. \tag{3.10}$$

Now let us consider the irreducible representations of the algebra. Clearly, the variable xy is block diagonal, and from (3.10) we see that it is proportional to the identity in each block, since z and σ are too. This is, $x_{i,i+1}y_{i+1,i}$ is proportional to the identity in the block corresponding to the node i, and it is equal to

$$x_{i,i+1}y_{i+1,i} = P_i w + \sum_k \tilde{Q}_k P_i \eta^{ik}/(1-\eta^k)\,. \tag{3.11}$$

So we can consider the quantity $r_i = \sum_k \tilde{Q}_k(z)\eta^{ik}/(1-\eta^k)$, and we have $x_{i,i+1}y_{i+1,i} = (w + r_i)P_i$.

If $x^n \neq 0$ and $y^n \neq 0$, then the product of the $x_{i,i+1}$ and $y_{i,i-1}$ is invertible, so each of them is invertible as well From the fact that $x_{i,i+1}y_{i+1,i}$ is proportional to the identity on the block i, it follows that all of the nodes have the same rank, this is, $x_{i,i+1}$ and $y_{i,i-1}$ establish isomorphisms between the neighboring nodes in the diagram. If the rank of each gauge group to be bigger than one, let's say N, then the unbroken gauge group is a diagonal $\mathrm{U}(N)$, since $x_{i,i+1}$ and $y_{i+1,i}$ are inverses of each other, and since x^n and y^n are proportional to the identity. Irreducibility implies that the unbroken gauge group is $\mathrm{U}(1)$, so $N_i = 1$ for all i. This is the condition for a single brane to be in the bulk.

Thus in the above, we can treat the $x_{i,i+1}$ and $y_{i+1,i}$ as numbers. We have then

$$\begin{aligned} u = x^n &= \prod_i x_{i,i+1}\,, \\ v = y^n &= \prod_i y_{i+1,i}\,, \\ x_i y_i &= w + r_i\,. \end{aligned} \tag{3.12}$$

And we have the relation

$$uv = \prod_i (w + r_i)\,, \tag{3.13}$$

where each r_i is polynomial in z. This equation is true in the full algebra $\mathcal{A}$. Verifying directly, as opposed to in each of the irreducible representations, takes a lot of manipulations with algebra, although it is easy to convince oneself from doing the algebra explicitly that it is indeed correct for the A_1 and A_2 singularities. Notice also that in the above equation $\sum r_i = 0$.

Here we see that the commutative geometry represents a family of resolutions of the A_{n-1} singularity, $uv = w^n$, parametrized by z. This is the original geometrically engineered version of the theory in [10, 11]. The analysis of these theories in terms of brane engineering and M-theory was worked in [22]–[25]

Indeed, it is possible to calculate explicitly the representations. We have already seen that $N_i = 1$. Notice that in the basis that diagonalizes the gauge group, we also have the σ diagonal and proportional to η^k on node k. Call these states $|k\rangle$. Now fix z and $u = x^n \neq 0$ let's say. Since $x^n \neq 0$, it is easy to check that $x^m|k\rangle \sim |k+m\rangle$, and we can choose the normalization factor to be constant by use of gauge transformations; so $x^m|k\rangle = \alpha^m|k+m\rangle$, with $\alpha^n = u$. So the orbit of the variable x acting on $|k\rangle$ is the full representation. The equations for the y_i are then linear difference equations

$$y_{i+1,i} - y_{i,i-1} = \alpha^{-1} Q'_i \tag{3.14}$$

so that all of the $y_{i,i+1}$ are determined once $y_{0,1}$ is known. The condition $\sum Q'_i(z) = 0$ is then required so that the linearly dependent equations above have a solution. Given $y_{1.0}$, one determines a unique value for w and $v = y^n$, and vice-versa; given a value of w, v which satisfies the constraints in the commutative algebra defined by u, v, w, z, there is a unique solution for $y_{1,0}$ which is compatible with it. Notice that in this normalization $y_{i,i-1}|k\rangle = \tilde{y}_{i,i-1}|k-1\rangle$, where the $\tilde{y}$ is a number.

Now, it is interesting to ask when can the representation be reducible. Indeed, from the form of the matrices it is clear that we need to be in a situation when applying x n times does not produce a full n-dimensional representation, and similarly for y. This is, we need $x^n = y^n = 0$, so there is at least one $x_{i,i+1} = 0$ and one $y_{j+1,j} = 0$; and we want these two to be associated to $i \neq j$, as otherwise the representation is of dimension n, and not reducible.

The conditions $x_{i,i+1} = 0$ and $y_{j+1,j} = 0$ mean that

$$x_{i,i+1} y_{i+1,i} = P_i(w + r_i) = 0\,, \qquad x_{j,j+1} y_{j+1,j} = P_j(w + r_j) = 0\,. \tag{3.15}$$

So we find that $w + r_i = 0$ and $w + r_j = 0$, since these are variables in the center and $P_{i,j} \neq 0$. This is, the representation is reducible exactly when two of the r_i are equal and $w = -r_i$. These are exactly the points that correspond to the singularities in the associated commutative geometry (3.13), when two of the roots in the product are equal $r_i = r_j$, $w = -r_i$ is minus the value of those equal roots, and when $u = v = 0$ too.

These smaller representations are fractional branes. Indeed, it is easy to see that the fractional brane representations correspond to roots of the extended Dynkin diagram of the $\hat{A}_{n-1}$ singularity, as they are related to k consecutive nodes having U(1) gauge group, and $n-k$ consecutive nodes having U(0) gauge group.

4. A non-toric non-orbifold singularity

Consider the quiver diagram given by one node and three adjoints, similar to the $\mathcal{N} = \triangle$ quiver diagram, but with a different superpotential

$$\mathrm{tr}(XYZ + ZYX) + \frac{\lambda}{3}(X^3 + Y^3 + Z^3) \tag{4.1}$$

with λ arbitrary. By the arguments of Leigh and Strassler [26], the $SU(N)$ gauge theory with this superpotential is conformal (one exploits the $\mathbb{Z}_3$ symmetry $X \to Y \to Z$) for some value of the gauge coupling. The quiver algebra has only one projector which is the identity, and the constraints are

$$XY + YX = \lambda Z^2\,, \qquad YZ + ZY = \lambda X^2\,, \qquad XZ + ZX = \lambda Y^2\,. \tag{4.2}$$

Consider the variables X^2, Y^2, Z^2.

It is easy to see that

$$[X^2, Y] = -\lambda[Z^2, X] = \lambda^2[Y^2, Z] = -\lambda^3[X^2, Y] \tag{4.3}$$

so that for generic λ we have that X^2 commutes with Y. One can produce a similar argument that shows that X^2 commutes with Z. And therefore $u = X^2, v = Y^2, w = Z^2$ are all variables in the center of the algebra.

It is easy to see that the algebra is finite dimensional over the center, as we can choose an order where X are before Y and Y are before Z by using the equations in the algebra; since the quantities X^2, Y^2, Z^2 on the right hand side of the equations commute with everything.

Consider now

$$[XYZ, X] = XY\lambda Y^2 - X\lambda Z^2 Z = \frac{\lambda}{2}[(XY^3 - Y^3X) - (XZ^3 - Z^3X) - (XX^3 - XX^3)]\,. \tag{4.4}$$

So both sides are written as a commutator with X. Thus $\gamma = XYZ - \frac{\lambda}{2}(X^3 - Y^3 + Z^3)$ commutes with X. Notice that this expression is invariant under the $\mathbb{Z}_3$ group that takes $X \to Y \to Z$, so it is in the center of the algebra; as it must also commute with Y and Z.

The algebra is of dimension 9 as a free module over the center of the algebra, and this corresponds to bulk representations in terms of 3×3 matrices.

It is now an easy matter to establish, using the same type of manipulations as in [21] that

$$\gamma^2 = auvw + b\lambda^2(u^3 + v^3 + w^3) \tag{4.5}$$

for some coefficients a, b. When $\lambda = 0$, this variety corresponds to a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, and the field theory describes the singularity with non-trivial discrete torsion. The variety above for generic λ has an isolated singularity at $\gamma = 0 = u = v = w$. The singularity does not seem to correspond to a toric singularity, as the equations satisfied by the variables are not given in terms of monomials that are equated to each other. Also, the right hand side of the equation can not be factored into linear terms for a generic $\lambda \neq 0$, as this would imply that the curve in u, v, w is singular inside $\mathbb{CP}^3$, so the singularity would not be isolated at the origin.

This singularity is obtained by complex structure deformation of a known singularity however. Notice that since the generic representation is of dimension 3, this is, we have a $U(3)$ gauge group for the bulk brane, and it splits at the origin into three trivial representations of the algebra where $Z, Y, X = 0$, each with $U(1)$ gauge group. The fractional branes are identical to one another.

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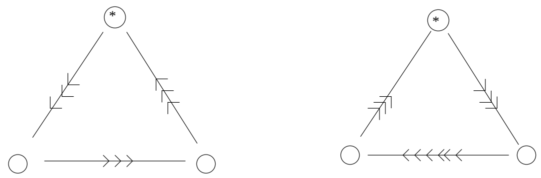

Figure 3: Two dual versions of the $\mathbb{C}^3/\mathbb{Z}_3$ quiver diagram, dualized at the marked node.

5. The $\mathbb{C}^3/\mathbb{Z}_3$ orbifold and Seiberg duality

We will now show another example that represents the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ after performing a duality on one of the nodes. One of the main reasons to understand this problem is from the knowledge that field theories which allow fractional branes give rise to cascading dualities [27] and geometric transitions. Although this is not the case for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold because there are no anomaly free configurations with fractional branes, one can still change the gauge couplings in the theory so that one of the nodes becomes infinitely strongly coupled. The continuation of the theory past this singular point is a dual theory. However, since this change is affected only by the Khaler structure of the target space, the holomorphic information should be invariant and one should be able to identify the geometric target space of both quiver diagrams. To set up the problem of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, let us first consider the orbifold quiver of $\mathbb{C}^3/\mathbb{Z}_3$ itself.

The orbifold quiver of $\mathbb{C}^3/\mathbb{Z}_3$ has three nodes V_{123}, as depicted in the left diagram of figure 3, and three sets of arrows $\phi^{1,2,3}_{i,i+1}$. The superpotential is given by

$$\mathrm{tr}(\epsilon_{ijk}\phi^i_{1,2}\phi^j_{2,3}\phi^k_{3,1})\,. \tag{5.1}$$

The field theory has an SU(3) global symmetry under which each of the $\phi_{i,i+1}$ transforms as a 3.

If we consider the quiver algebra, and the fields ϕ^i defined as follows

$$\phi^i = \phi^i_{1,2} + \phi^i_{2,3} + \phi^i_{3,1} \tag{5.2}$$

then the superpotential equations of motion reduce to

$$[\phi^i, \phi^j] = 0\,. \tag{5.3}$$

Thus the algebra of the ϕ^i is commutative and one can think of them as generators of $\mathbb{C}^3$, however, to be elements of the center of the quiver algebra, they need to commute with the projectors associated to the vertices as well. These elements of the center are generated by elements of the form

$$z^{ijk} = \phi^i\phi^j\phi^k \tag{5.4}$$

and one can see that they transform in the 10 of the global SU(3) (totally symmetric in three SU(3) indices), moreover these are exactly the coordinates that are invariant under the $\mathbb{Z}^3$ action on $\mathbb{C}^3$ that sends $\phi^i \to \eta\phi^i$, where $\eta^3 = 1$. So the algebra of the center describes the orbifold space $\mathbb{C}^3/\mathbb{Z}_3$.

Now take the quiver diagram, and perform Seiberg duality [28] on one of it's nodes. The resulting quiver diagram is depicted on the right of figure 3. This does not change the moduli space of vacua, so one should be able to identify V from any of it's dual versions. The duality for this particular case was described in [11], and gives rise to a new quiver diagram where there is no apparent orbifold point at which there is an extra $\mathbb{Z}_3$ symmetry of rotating the nodes clockwise. The duality is described in terms of mutations of bundles, and they correspond to particular changes of basis for D-brane charge. The description changes one exceptional collection of bundles that describes the D-branes in the large volume limit of the resolution for another exceptional collection. More general dualities related to toric singularities have been analyzed and proposed in [6, 7, 8, 29]. Since it is not clear that after performing Seiberg duality the center of the new quiver algebra has any relation with the center of the original quiver algebra, it is worth checking that the centers give rise to the same commutative geometry. Later we will argue why this should be the case always.

The quiver diagram to consider is as shown in the figure. We have field $\chi^{\alpha\beta}$, ϕ_α, ξ_β which transform in the 6, $\bar{3}$, $\bar{3}$ representations of the SU(3) global symmetry, and we are given the superpotential

$$W = \mathrm{tr}(\chi^{\alpha\beta}\phi_\alpha\xi_\beta)\,. \tag{5.5}$$

We have three projectors $P_{1,2,3}$, and the non-trivial constraints

$$\phi_\alpha\xi_\beta + \phi_\beta\xi_\alpha = 0\,, \qquad \chi^{\alpha\beta}\phi_\beta = 0\,, \qquad \xi_\beta\chi^{\alpha\beta} = 0\,. \tag{5.6}$$

For a variable to be in the center, it must commute with the three projectors, and since we have a conformal field theory at the origin in moduli space let us pick one such of minimal conformal weight which is not the identity. It must be of the form

$$a_1\chi\xi\phi + a_2\phi\chi\xi + a_3\xi\phi\chi\,. \tag{5.7}$$

This expression we decompose into irreducible representations of the SU(3) global symmetry, and we want to check that there is a 10 in the center of the algebra.

Notice that there is only one 10-dimensional representation of SU(3) in the product

$$6\otimes\bar{3}\otimes\bar{3} = 1\oplus 8\oplus 8\oplus 27\oplus 10 \tag{5.8}$$

which is antisymmetric in the two $\bar{3}$ indices. From here it is clear that the condition of being in the center will fix the ratios of the coefficients $a_1 : a_2 : a_3$.

It is clear that the above expression commutes automatically with $P_{1,2,3}$, so we only need to check the commutation relations with ϕ, ξ, χ. Indeed, we only need to do this for only one function in the 10 (the highest weight state), as one can generate any other function in the 10 by using SU(3) rotations on a single state. The functions are more explicitly given by

$$f^{\alpha\beta\epsilon} = \epsilon^{\gamma\delta[\epsilon}(a_1\chi^{\alpha\beta]}\xi_\gamma\phi_\delta + a_2\phi_\delta\chi^{\alpha\beta]}\xi_\gamma + a_3\xi_\gamma\phi_\delta\chi^{\alpha\beta]})\,, \tag{5.9}$$

where repeated indices are contracted and the square parenthesis indicate that we take a totally symmetric representation in three indices ϵ, α, β. In particular, we will take $\epsilon = \alpha = \beta = 1$ to check the commutation relations, with arbitrary ϕ, χ, ξ.

Consider $[f^{111}, \phi_\alpha]$. This is equal to

$$a_2(\phi_2\chi^{11}\xi_3 - \phi_3\chi^{11}\xi_2)\phi_\alpha - a_1\phi_\alpha\chi^{11}(2\xi_3\phi_2)\,. \tag{5.10}$$

Now we have to consider the cases $\alpha = 3, 2, 1$ separately.

For $\alpha = 3$, we have

$$\begin{aligned} a_2(\phi_2\chi^{11}\xi_3 - \phi_3\chi^{11}\xi_2)\phi_3 - a_1\phi_3\chi^{11}(2\xi_3\phi_2) &= a_2(\phi_2\chi^{11}\xi_3\phi_3) - (a_2 - 2a_1)\phi_3\chi^{11}\xi_2\phi_3 \\ &= (2a_1 - a_2)\phi_3\chi^{11}\xi_2\phi_3\,, \end{aligned} \tag{5.11}$$

where we have used the superpotential equations repeatedly as $\xi_3\phi_3 = 0$, and $\xi_2\phi_3 = -\xi_3\phi_2$. This expression vanishes if $a_2 = 2a_1$. One can check that the same result is obtained from commuting with ϕ_2.

The case $\alpha = 1$ is more involved algebraically. Here we get

$$\begin{aligned} a_2(\phi_2\chi^{11}\xi_3 - \phi_3\chi^{11}\xi_2)\phi_1 - a_1\phi_1\chi^{11}(2\xi_3\phi_2) &= -2a_1(\phi_2\chi^{11}\xi_1\phi_3 - \phi_3\chi^{11}\xi_1\phi_2) - \\ &\quad - a_1\phi_1\chi^{11}(2\xi_3\phi_2)\,. \end{aligned} \tag{5.12}$$

Now we need to use the superpotential relations that involve $\chi^{ij}\xi_j = \phi_j\chi^{ij} = 0$, so we obtain

$$-2a_1(-\phi_2\chi^{22}\xi_2\phi_3 + \phi_3\chi^{33}\xi_3\phi_2) + 2a_1(\phi_2\chi^{22} + \phi_3\chi^{33})(\xi_3\phi_2) \tag{5.13}$$

and we see that the terms cancel exactly.

The computations for commuting with ξ are similar, and one obtains that $a_2 = 2a_3$ as well. in this way, we have fixed the ratio $a_1 : a_2 : a_3 = 1 : 2 : 1$; and the result commutes with both ϕ and ξ. Last of all, one needs to check that the expression commutes with χ^{ij} as well.

It is easy to check the commutation relations with χ^{11}, but for the other χ^{ij} we need to convert the indices that are not equal to 1 to indices that are equal to 1 by applying the superpotential relations. For example

$$\begin{aligned} a_1\chi^{12}\xi_2\phi_3\chi^{11} - a_1\chi^{11}\xi_2\phi_3\chi^{12} &= -a_1\chi^{11}\xi_1\phi_3\chi^{11} + a_1\chi^{11}\xi_3\phi_2\chi^{12} \\ &= a_1\chi^{11}\xi_3\phi_1\chi^{11} - a_1\chi^{11}\xi_3\phi_1\chi^{11} \end{aligned} \tag{5.14}$$

which vanishes. The verification of the commutation relations for the remaining $\chi^{\alpha\beta}$ are left to the reader as an exercise.

The upshot of the calculation is that there are variables in the 10-dimensional representation of SU(3) which are in the center of the algebra. One needs to verify that these also satisfy the appropriate constraints at the $(f)^2$ level. That is easier to do by calculating the moduli space of irreducible representations of the algebra away from the singularity, but this is exactly the type of information that is invariant under Seiberg duality, so we will not repeat the calculation of the moduli space here.

Now, one might ask why is the center invariant under Seiberg duality transformations? The answer lies in representation theory. For a single point like brane, variables in the center are gauge invariant since they are proportional to the identity. This is, they can be written in terms of their trace multiplied by the identity operator on the quiver algebra. Since these variables are gauge invariant, they should be respected by duality transformations; as well as the relations that they satisfy, since this is the information that determines the moduli space of vacua of the supersymmetric theory.

Similarly, given a collection of N separated branes, these variables will be block diagonal on the representation and then the eigenvalues of the matrices f can be recovered by taking traces $\mathrm{tr}(f^k)$. In essence, these variables in the center can be reconstructed from gauge invariant operators on any brane configuration; and the gauge invariant variables are independent of which dual representation of the theory one chooses.

6. Conclusion

We have seen how given a quiver diagram with some choice of superpotential we can associate to it an algebro-geometric space V which is such that the moduli space of D-branes is essentially the symmetric space $\mathrm{Sym}^N(V)$, except for the singularities of V where fractional branes can appear. We have shown many examples of the procedure where we find that V is indeed a complex variety of dimension 3.

The construction is based on finding the center of a non-commutative meson algebra associated to the quiver field theory. The relations in the quiver algebra include the superpotential equations of motion and therefore encode the moduli space problem for D-branes. The space V is the variety associated to the center of this quiver algebra. One expects that the reverse geometric engineering is sensible only if the algebra of the quiver is finitely generated as a module over the center, which seems to be a severe restriction on the quiver diagram and the superpotential.

On the cases when it exists, the space V is invariant under Seiberg dualities, so the construction of V is unambiguous for different dual descriptions of the same theory.

It would be very interesting if one could go beyond the $\mathrm{U}(N)$ gauge groups and include orientifold constructions as well. However, this is more subtle as one needs to understand the holomorphic properties of an algebra of unoriented strings before one can hope to develop these techniques further in this direction.

One can also hope to build new singularities from interesting quiver theories and perhaps this problem can result more tractable to provide a classification of singularities in complex dimension 3.

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